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# Controlling potential traps for filtering solitons in Bose–Einstein condensates

*R. Fedele*[1)], *P. K. Shukla*[2)], *S. De Nicola, M. A. Man'ko*[3)], *V. I. Man'ko, F. S. Cataliotti*[4)]

*Dipartimento di Scienze Fisiche and INFN, Università Federico II di Napoli, I-80126 Napoli, Italy*

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We present a controlling potential method for solving the three-dimensional Gross–Pitaevskii equation (GPE), which governs the nonlinear dynamics of Bose–Einstein condensates (BECs) in an inhomogeneous potential trap. Our method allows one to construct ground and excited matter wave states whose longitudinal profiles can have bright solitons. This method provides the confining potential that filters and controls localized BECs. Moreover, it is predicted that, while the BEC longitudinal soliton profile is controlled and kept unchanged, the transverse profile may exhibit oscillatory breathers (unmatched case) or moving as a rigid body in the form of either coherent states (performing the Lissajous figures) or a Schrödinger cat state (matched case).

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It is well known that the dynamics of Bose–Einstein condensates (BECs) in condensed matters is governed by the Gross–Pitaevskii equation (GPE) [1]. Stationary solutions of the latter in one space dimension can be cast in the form of solitons [2]. The formation of bright and dark/grey solitons is attributed to the attractive as well as repulsive inter-atomic interaction and that they were created experimentally in elongated BECs [3].

We consider here the multidimensional GPE with confining potentials for BECs, and present a new analytical method for filtering different kinds of solitons by controlling the confining potential. This idea could be realized by recently developed techniques involving lithographically fabricated circuit patterns which provide electromagnetic guides and microtraps for ultracold neutral systems of atoms in BEC experiments [4]. Alternatively, the use of optically induced potentials is also extremely versatile for producing of "exotic" potentials [5].

We obtain exact controlled 3D solutions for the ground and excited soliton states of BECs.

The dynamics of BECs in a spatially nonuniform confining potential well is governed by the GPE [1]

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m_a}\nabla^2\Psi(\mathbf{r},t) + U(\mathbf{r},t,|\Psi(\mathbf{r},t)|^2)\Psi, \tag{1}$$

where $\hbar$ is the Planck constant divided by $2\pi$, $\Psi(\mathbf{r},t)$ is the macroscopic wave function of the condensate, $m_a$ is the atomic mass, $U(\mathbf{r},t,|\Psi(\mathbf{r},t)|^2) = V_{\text{ext}}(\mathbf{r},t) + gN|\Psi(\mathbf{r},t)|^2$ , $V_{\text{ext}}(\mathbf{r},t)$ is the external confining potential for BECs, the coupling constant $g$ is related to the short range scattering (s-wave) length $a$ representing the interactions between atomic particles, namely, $g = 4\pi\hbar a/m_a$, and $N$ is the number of atoms. The short range scattering length $a$ of atoms can be either positive or negative giving rise to either attractive or repulsive forces.

We consider a generic external potential composed of two parts, viz. $V_{\text{ext}}(\mathbf{r},t) = V_\perp(\mathbf{r}_\perp,t) + V(z,t)$. We look for a solution of (1) in the form $\Psi(\mathbf{r},z,t) = \Psi_\perp(\mathbf{r}_\perp,t)\Psi_z(z,t)$. Hence, Eq. (1) can be decomposed as

$$i\hbar\frac{\partial\Psi_\perp}{\partial t} = -\frac{\hbar^2}{2m_a}\nabla_\perp^2\Psi_\perp + V_\perp(\mathbf{r}_\perp,t)\Psi_\perp, \tag{2}$$

$$i\hbar\frac{\partial\Psi_z}{\partial t} = -\frac{\hbar^2}{2m_a}\frac{\partial^2\Psi_z}{\partial z^2} + g_{1\text{D}}N|\Psi_z|^2\Psi_z + V(z,t)\Psi_z\,, \tag{3}$$

where $g_{1\text{D}} = g\int|\Psi_\perp|^4 d^2r_\perp$. We see that the transverse and longitudinal equations are exactly decoupled, the former being a linear equation and the latter being the 1D cubic nonlinear Schrödinger equation (cNLSE) whose nonlinear coupling coefficient depends on the shape of the transverse BEC's density profile. Note also that $g_{1\text{D}}$ is, in principle, function of $t$ ($g_{1\text{D}} = g_{1\text{D}}(t)$).

We assume that the transverse external potential is quadratic, i.e. $V_\perp(x,y,t) \equiv m_a\left[\omega_x^2(t)x^2 + \omega_y^2(t)y^2\right]/2$. Thus, Eq. (2) admits, in rectangular coordinates, the following complete set of normalized Hermite-Gauss modes: $\Psi_{\perp nm}(x,y,t) = \Psi_{xn}(x,t)\Psi_{ym}(y,t)$, where

[1)] e-mail: renato.fedele@na.infn.it
[2)] On leave from Ruhr-Universitat Bochum, Germany.
[3)] On leave from P. M. Lebedev Physical Institute, Moscow.
[4)] On leave from University of Catania, Italy.

$$\Psi_{jk}(j,t) = H_k\left[j/\sqrt{2}\sigma_j(t)\right] \times \frac{\exp\left[-j^2/4\sigma_j^2(t) + im_a\gamma_j(t)j^2/2\hbar + i\phi_{jk}(t)\right]}{\left[2\pi\sigma_j^2(t)2^{2k}(k!)^2\right]^{1/4}}, \quad (4)$$

with $j = x, y$ and $k = 0, 1, 2, 3, \ldots$ The quantities $\sigma_j(t)$, $\gamma_j(t)$ and $\phi_{jk}$ satisfy the following system of equations

$$d^2\sigma_j/dt^2 + \omega_j^2(t)\sigma_j - \hbar^2/4m_a^2\sigma_j^3 = 0, \quad (5)$$

$\gamma_j(t) = (1/\sigma_j(t))(d\sigma_j(t)/dt)$, $\phi_{jk}(t) = (2k+1)\,\phi_{j0}(t)$, and $d\phi_{j0}(t)/dt = -\hbar/4m_a\sigma_j^2(t)$. Since Eq. (2) is linear, in principle, an arbitrary normalized solution $\Psi_\perp$ can be expressed as a linear combination of the above $\Psi_{\perp nm}$, i.e. $\Psi_\perp = \Sigma_{nm} c_{nm}\Psi_{\perp nm}$. Note that $\Psi_{j0}(j,t)$ is a purely Gaussian time-dependent fundamental mode and $\sigma_j(t)$ is its rms $\sqrt{\langle j^2\rangle}$ . Consequently, the transverse effective spot size of the condensate can be defined as $\sigma_\perp(t) \equiv \sqrt{\sigma_x^2(t) + \sigma_y^2(t)}$. Equation (5) (Pinney equation) describes the envelope oscillations of the condensate along the $j$-th transverse direction within the quadratic potential well $V_j(j,t) = m_a\omega_j(t)j^2/2$.

For $\omega_j$ independent of time, Eq. (5) can be easily integrated. For the initial conditions: $\sigma_{j0} \equiv \sigma_j(t=0)$ and $\gamma_{j0} \equiv \gamma_j(t=0)$, we have for the BEC transverse envelope motion

$$\sigma_j(t) = \sigma_{j0}\left[\left(\cos\omega_j t + \frac{\gamma_{j0}}{\omega_j}\sin\omega_j t\right)^2 + \frac{\sigma_j^{*4}}{\sigma_{j0}^4}\sin^2\omega_j t\right]^{1/2}, \quad (6)$$

where $\sigma_j^* = \sqrt{\hbar/m_a\omega_j}$. From Eq. (6) it is evident that, if $\gamma_{j0} \neq 0$, BECs execute envelope oscillations (breathers). If $\gamma_{j0} = 0$, the $j$-th transverse BEC state is described by a wave function (given for each integer $k$ by Eq. (4) ) whose rms does not change in time (stationary state) when $\sigma_{j0} = \sigma_j^*$ (matched case), or it executes oscillatory breathers when $\sigma_{j0} \neq \sigma_j^*$ (unmatched case). In particular, for $k = 0$, this $j$-th transverse state is described by the 1D harmonic oscillator ground state, which is purely Gaussian and has the characteristic to be the simplest coherent state. An overcomplete set of coherent states in such a harmonic potential well can be obtained just by shifting the center of the Gaussian state by a time-dependent vectorial shift, say $\mathbf{r}_{\perp 0}(t) = (x_0(t), y_0(t))$, with respect to the minimum of the potential well obeying the classical harmonic oscillator motion.

We seek a solution of Eq. (3) in the form $\Psi_z(z,t) = F(z,t)\exp(-iE_a t/\hbar)$, where $E_a$ is a real number and $F$ is a complex function (the solution of a cNLSE)

$$i\hbar\partial F/\partial t = -(\hbar^2/2m)\partial^2 F/\partial z^2 + \beta|F|^2 F. \quad (7)$$

Here $\beta$ is a real constant. Consequently, we have to impose the following condition for the potential: $V(z,t) = [\beta - g_{1\mathrm{D}}(t)N]\,|F|^2 + E_a$. Equation (7) admits bright (dark/grey) soliton solutions for $\beta < 0$ ($\beta > 0$). The main property of these solutions is that they are stationary profile solution (nonlinear traveling waves), namely, $F(z,t) = \mathcal{F}(\xi)\exp(-iE_s t/\hbar)$, where $\xi \equiv z - u_0 t$, $u_0$ and $E_s$ are arbitrary constants representing the soliton speed and nonlinear frequency shift, respectively. For $\beta < 0$ the bright ($E_a = 0$) standing (no travelling) envelope soliton solution of Eq. (7) is $F(z,t) = (1/\sqrt{2l_z})\ \mathrm{sech}\,(z/l_z)\ \exp(-iEt/\hbar)$, where $E = \int F^*\hat{H}_z F dz$ is a negative arbitrary constant, which plays the role of energy eigenvalue associated with the "nonlinear longitudinal eigenstate" of Eq. (7) and characterizes a continuous spectrum of the soliton energy.

In view of the above analysis, we realize that a method for filtering longitudinal bright soliton states of BECs can be established. Let us suppose that we want to produce a bright soliton with amplitude $F_M$ and width $l_z$. First we find from $F(z,t)$ the corresponding value of $\beta$ and $E$, i.e. $\beta = -\hbar^2/m_a l_z^2 F_M^2$, $E = -\hbar^2/2m_a l_z^2$. The latter shows a direct correspondence between the BEC longitudinal width $l_z$ (soliton width) and its longitudinal energy $E$ (soliton energy). So one could also characterize the soliton continuous spec trum in terms of $l_z$. We then design, using the techniques described in [4, 5], an external potential well $V(z,t) = -(\hbar^2/ml_z^2)\left[1 + (mNl_z/2\hbar^2)\ g_{1\mathrm{D}}(t)\right]\ \mathrm{sech}^2(z/l_z)$. Note that the external longitudinal potential well is time-modulated through the factor $g_{1\mathrm{D}}$ which, in turn, depends on time due to time variation of the transverse rms $\sigma_j(t)$. Assuming that, by virtue of Eq. (4), $\Psi_\perp = \Psi_{\perp nm}$ we have $g_{1\mathrm{D}}^{(nm)}(t) \equiv g\int|\Psi_{\perp nm}(x,y,t)|^4 d^2r_\perp$, which can be cast as $g\ \delta_n\delta_m/2\pi^2\sigma_x(t)\sigma_y(t)$, where $\delta_k = (1/2^{2k}(k!)^2) \times \int_{-\infty}^{\infty}\ \exp\left(-2\xi^2\right)\left[H_k(\xi)\right]^4 d\xi$. Let us now discuss the possibilities to use different modes as solutions to the linear transverse evolution equation describing the BEC state related to the factorized form of the BEC wave function. Having a solution $\Psi_\perp(x,y,t)$, one has controlling tools for the longitudinal soliton solution of the GPE. These tools are reduced to time-dependent variable $g_{1\mathrm{D}}$. When each $\omega_j$ does not depend on time, $\sigma_j(t)$ is given by Eq. (6). Consequently, we have

$$V(z,t) = -\frac{\hbar^2}{m_a l_z^2}\left[1 + \frac{m_a N l_z g\delta_n\delta_m}{4\pi^2\hbar^2\sigma_x(t)\sigma_y(t)}\right]\mathrm{sech}^2\frac{z}{l_z}. \quad (8)$$

By virtue of (4) it is easy to see that the transverse BEC probability density of matter waves $|\Psi_{\perp nm}(x,y,t)|^2$ is affected by oscillatory breathers. In general, due to the linearity of Eq. (2), an arbitrary transverse BEC state

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Fig.1. Cross sections of the BEC transverse probability density $|\Psi_\perp|^2$ at $z = 0$ and different times: $t_1 = 1.57/\omega_x$ (a), $t_2 = 3.14/\omega_x$ (b), $t_3 = 4.71/\omega_x$ (c), and $t_4 = 6.28/\omega_x$ (d). The three groups of panels are plotted for different values of mode indices: $n = 0$, $m = 0$; $n = 1$, $m = 1$; $n = 2$, $m = 2$. Calculations refer to $\omega \equiv \omega_y/\omega_x = 0.5$, $\sigma_x^*/l_z = 0.5$, $\sigma_y^*/l_z = \sigma_x^*/l_z\sqrt{\omega} = 1/\sqrt{2}$. Normalized transverse widths are $\sigma_{x0}/l_z = 0.5$ and $\sigma_{y0}/l_z = 1$; $\gamma_{x0}/\omega_x = 0.5$ and $\gamma_{y0}/\omega_y = 1$

$\Psi_\perp$, which can be expressed as a superposition of the Hilbert space base $\{\Psi_{\perp nm}\}$, should exhibit similar oscillatory breathers as well. It turns out that in order to filter and control a longitudinal bright soliton profile of BEC while preserving its width, one has to impose an external potential well given by (8), which takes into account the transverse BEC breathers. A discussion of some special cases is in order. First, we consider the case of stationary transverse states (eigenstates of the transverse Hamiltonian $\hat{H}_\perp$). Here the rms of the modes associated with the $j$-th transverse direction satisfies the initial condition in the matched case ($\sigma_{j0} = \sigma_j^*$ with $\gamma_{j0} = 0$) and, consequently, does not change in time (no spread variation of the transverse BEC state). Thus, the external longitudinal potential well is reduced to the following time-independent form $V(z) = -(\hbar^2/m_a l_z^2)(1 + m_a N l_z g \delta_n \delta_m / 4\pi^2 \hbar^2 \sigma_x^* \sigma_y^*)\mathrm{sech}^2(z/l_z)$, which shows that, in order to preserve the soliton distribution width (no spread variations), the potential controlling system has not to account for breather effects. It follows that the corresponding controlled 3D BEC state is $\Psi_{nml_z}(x,y,z,t) = \psi_{nml_z}(x,y,z)\exp[iE_{nm}/\hbar]$, $E_{nm} = (n+1/2)\hbar\omega_x + (m+1/2)\hbar\omega_y - \hbar^2/2m_a l_z^2$,

$$\psi_{nml_z}(x,y,z,t) = \frac{\exp\left[-\left(x^2/4\sigma_x^{*2}\right) - \left(y^2/4\sigma_y^{*2}\right)\right]}{\left[4\pi\sigma_x^*\,\sigma_y^*\,l_z\,2^{n+m}n!m!\right]^{1/2}}$$

$$\times H_n\left(x/\sqrt{2}\sigma_x^*\right) H_m\left(y/\sqrt{2}\sigma_y^*\right)\ \mathrm{sech}\,(z/l_z)\,, \qquad (9)$$

where the subscript $l_z$ accounts for the continuity of the longitudinal energy spectrum. On the contrary, the transverse energy is characterized by a discrete spectrum, i.e. $\langle \hat{H}_{\perp nm}\rangle = (n+1/2)\hbar\omega_x + (m+1/2)\hbar\omega_y$.

Figure 1 clearly shows the oscillatory breathers of both the BEC transverse density distributions and the temporal controlling potential function, due to the transverse rms (spot size) oscillations. While this BEC "respiration" takes place, the soliton-like BEC longitudinal profile remains unchanged. For $n = m = 0$, the 2D transverse BEC cross section (transverse spot) is an ellipse whose shape changes progressively as the ratio $\sigma_x(t)/\sigma_y(t)$ changes in time. In particular, this ratio, which initially is $< 1$, reaches 1, and subsequently becomes $> 1$. For $n$ and/or $m$ greater than 1, the transverse spot are more than 1. For increasing values of $n$ and $m$ (higher modes), the numbers of transverse spots increase correspondingly and the time evolution of all the transverse spots shows the oscillatory breathers.

Second, we consider the case where the transverse BEC profile corresponds to a 2D coherent state. Consequently, while the soliton-like longitudinal BEC profile is kept by potential controlling, the transverse one is a bi-Gaussian with a moving centroid (guiding center)

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Fig.2. Centroid motion of the coherent wavepacket performing Lissajous figures (left panel) and corresponding BEC transverse probability density $|\Psi_\perp|^2$ at $z = 0$ (right panel) for the ratio $\omega_y/\omega_x$ equal to $1.5, 1.0$, and $0.5$, respectively (from top to bottom). The $x$ and $y$ components of the maximum amplitude of the centroid motion, normalized with respect to $l_z$, are 5 and 10, respectively. The relative phase between them is 1.57 rad and $\sigma_x^*/l_z = 0.5$, $\sigma_y^*/l_z = \sigma_x^*/l_z\sqrt{\omega_y/\omega_x}$. The ellipse eccentricity $\sigma_x^*/\sigma_y^*$ is governed by $\sqrt{\omega_y/\omega_x}$

whose coordinates, say $x_0(t)$ and $y_0(t)$, satisfy the classical harmonic oscillator equations: $d^2 j_0/dt^2 + \omega_j j_0 = 0$, $p_{0j} = -m_a dj_0/dt$. Thus, the Gaussian transverse BEC distribution moves, as its centroid, around in the transverse $x - y$ plane as a rigid body preserving shape and rms (no spread variations). This motion is the composition of the rigid oscillations along each transverse direction of each Gaussian distribution (preserving shape and rms as well) according to the corresponding centroid oscillation. The phase relation between these two oscillations should be imposed by the initial conditions. In principle, the transverse BEC centroid describes the Lissajous figures. While the transverse motion takes place, the shape of the longitudinal soliton distribution is preserved together with its width (no spread variations). Figure 2 shows the Lissajous figures described by the "rigid" motion of a BEC coherent state distribution in the $x - y$ plane, which does not exhibit any oscillatory breathers.

The third special case is the one where $\Psi_\perp$ is a Schrödinger cat state. In the Dirac formalism, we can construct such a state by taking, for the $j$-th transverse direction, the superposition $(|\alpha_j\rangle \pm |-\alpha_j\rangle)/\sqrt{2}$, called even (sign "+") or odd (sign "-") coherent state, where $\alpha_j(t) = j_0(t)/2\sigma_j^* + \sigma_j^* p_0(t)/\hbar$. This combination is also known as Schrödinger cat (associated with the $j$-th transverse direction). In order to have a 2D Schrödinger cat, one may construct the combination $(|\vec{\alpha}\rangle \pm |-\vec{\alpha}\rangle)/\sqrt{2}$, where $\vec{\alpha}$ is the complex vector: $\vec{\alpha} = \hat{x}\alpha_x + \hat{y}\alpha_y$. Figure 3 shows density plots

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Fig.3. Cross sections of the BEC transverse probability density $|\Psi_\perp|^2$ associated with even coherent state (Schrödinger cat) at $z = 0$ and different times: $t_1 = 6.0/\omega_x$, $t_2 = 6.3/\omega_x$, $t_3 = 6.5/\omega_x$, and $t_4 = 6.7/\omega_x$. At time $t_2$ the Schrödinger cat state shows a marked interference effect as it can be seen by the formation of a clear interference fringe pattern. The $x$ and $y$ components of the maximum amplitude of the centroid motion, normalized with respect to $l_z$, are 5 and 5, respectively. The relative phase between them is 1.57 rad and $\sigma_x^*/l_z = 0.5$, $\sigma_y^*/l_z = \sigma_x^*/l_z\sqrt{\omega_y/\omega_x}$ ($\omega_y/\omega_x$ is put 1/5)

in the $x/\sigma_x^* - y/\sigma_y^*$ plane of $|\Psi_\perp(x,y)|^2$ corresponding to a transverse BEC Schrödinger cat state at different times.

In conclusion, we have presented a controlling potential method for solving the 3D GPE which governs the nonlinear dynamics of BECs in nonuniform confining potentials. This novel approach seems to be possible thanks to the increasingly employed techniques to produce almost arbitrary potential trap shapes, by using suitably tapered wires as in the microtraps techniques as well as by using suitably shaped laser beams as in the optical potential techniques. We have shown that stationary bright solitons with finite transverse extent are filtered and controlled by a proper potential trap which, in turn, is selected on the basis of the shape of soliton excited in BECs, and therefore $\propto \mathrm{sech}^2(z/l_z)$. The present result shows that by controlling the longitudinal potential it is possible to control the transverse characteristics of a BEC. This is of crucial importance for the possible realization of atomic waveguides especially in the framework of lithographic microtraps. We would like to point out that the present solitonic solutions are stable, i.e. a small deviation from the potential $V(z,t)$ will not result in a destruction of the soliton. A more complete stability analysis will be the subject of future work.

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