New localized solutions of the nonlinear field equations

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The interactions described by invariants which are linear in the first spatial derivatives (Lifshitz invariants) stabilize two- and three-dimensional localized states. The interaction force at large distances is determined for two-dimensional localized states (vortices). © *1995 American Institute of Physics.*

1. Localized solutions (solitons) of nonlinear equations in mathematical physics have been for many years an object of research in the theory of the condensed state, in the elementary-particle and nuclear physics, in astrophysics and cosmology, and in biophysics.[1] The interest in such solutions in elementary-particle physics is connected with attempts to construct a structural theory in which the elementary particles are localized solutions of nonlinear field equations and the physical fields are described by the asymptotic solutions.[1] Such a theory would make it possible to remove the particle–field duality and to eliminate other inconsistencies of theories based on the concept of point particles.

The advances made in the modern theory of solitons mainly involve one-dimensional systems. Moreover, it has been shown for multidimensional localized states that for many physical field models such solutions are unstable.[1–4] For a long time, Abrikosov vortices[5] in superconductors have been the only example of multidimensional localized stationary states in physical systems. However, in Refs. 6 –8 it was shown that in magnetic materials without an inversion center the invariants which are linear in the first spatial derivatives (Lifshitz invariants) stabilize the two-dimensional localized states — magnetic vortices. The equations describing these nonuniform states are much simpler than the corresponding equations for flux lines in superconductors. Vortex structures in systems with Lifshitz invariants are therefore a convenient object for modeling the "corpuscular" properties of localized states.

Besides magnetic materials,[9,10] interactions described by invariants which are linear in the first spatial derivatives also occur in certain classes of ferroelectrics, liquid crystals, and other systems.[11,12]

In the present paper we calculate the interaction force between two magnetic vortices. It is shown that the interactions described by the Lifshitz invariants stabilize two- and three-dimensional localized states.

2. We shall analyze the field of the unit vector $\mathbf{n}=(\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta)$ with the interaction functional

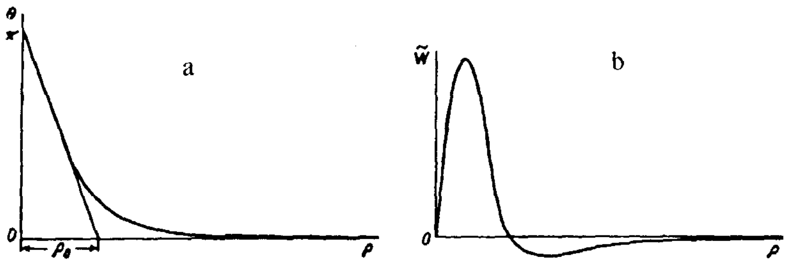

FIG. 1.

$$W=\int w d\mathbf{x}=\int\left[\left(\frac{\partial\mathbf{n}}{\partial x_i}\right)^2+\frac{4k}{\pi}\mathbf{n}\,\mathrm{curl}\,\mathbf{n}+(1-n_z^2)+2h(1-n_z)\right]d\mathbf{x}. \tag{1}$$

The first term describes the stiffness of the system and the second term is a possible form of the Lifshitz invariant. The functional (1) describes the energy of a cubic ferromagnet without an inversion center[10] and with induced uniaxial anisotropy in a magnetic field h. The choice of the functional (1) is dictated by the fact that here we must use some results obtained in Refs. 7 and 8 for magnetic vortices. As shown below, however, the nonuniform states we are studying arise in systems with an interaction functional of a quite general form.

For $|k|<1$ and $h>0$, the minimum of the functional (1) corresponds to a uniform state with $\mathbf{n}_0=(0,0,1)$. However, this solution does not exhaust all stationary states of the field. Following Refs. 6–8, it can be shown that among the metastable excitations of this field there exist axisymmetric states $\psi(\varphi)$ and $\theta(\rho)$ (vortices) which are uniform with respect to z. Here we introduce cylindrical coordinates $\mathbf{x}=(\rho\cos\varphi, \rho\sin\varphi, z)$ for the spatial variable.

For $0<k<1$ the solutions $\psi(\varphi)=\varphi-\pi/2$ satisfy the Euler equations for the functional (1), and $\theta(\rho)$ is determined from the equation

$$\frac{d^2\theta}{d\rho^2}+\frac{1}{\rho}\frac{d\theta}{d\rho}-\frac{1}{\rho^2}\sin\theta\cos\theta+\frac{4k}{\pi}\frac{\sin^2\theta}{\rho}-\sin\theta\cos\theta-h\sin\theta=0 \tag{2}$$

with the boundary conditions $\theta(0)=\pi$ and $\theta(\infty)=0$.

In Refs. 7 and 8 the equilibrium parameters and the limits of existence of magnetic vortices were determined by integrating Eq. (2) numerically. Figure 1 shows the characteristic functions $\theta(\rho)$ (a) and the "linear" density $\tilde{w}(\rho)$ (b) of the equilibrium energy

$$\tilde{W}=\int_0^\infty \tilde{w}(\rho)d\rho.$$

In the central part (the nucleus) the energy density of the vortex is higher than the energy density of the ground state $\mathbf{n}_0$ (vacuum) and becomes negative in the peripheral region (see Fig. 1b). For $\rho\gg 1$ we have

$$\theta(\rho)=A\exp(-\rho\sqrt{1+h}). \tag{3}$$

In the region under investigation the vortices are strongly localized excitations of the field with positive (relative to the vacuum) energy (see Fig. 1). The nucleus in which most of the energy of a vortex is concentrated can be viewed as a two-dimensional "elementary particle." The quantity

$$\rho_0 = \pi(d\theta/d\rho)^{-1}_{\rho=0}$$

can serve as the size of the particle. The weak perturbation of the vacuum in the region $\rho \gg 1$ (3) is the "field" generated by the particle. The character of the interaction of the particles is determined by the direction of rotation of θ: Particles with like directions of rotation repel one another and particles with opposing directions of rotation attract one another. The arguments here are the same as in the analysis of the interaction of planar Bloch domain walls in a ferromagnet.[13] The topological charge[1]

$$Q = \frac{1}{4\pi}\int \sin\theta(\mathbf{r})\, d\theta(\mathbf{r})\, d\psi(\mathbf{r}),$$

which assumes the value 1 or -1, depending on the direction of rotation of θ in the vortices, plays the role of an electric charge.

To calculate the interaction force between parallel flux lines separated by a distance $r \gg 1$, we employ an expression for the change in the momentum of the field $\mathbf{F}$ in terms of the components of the stress tensor T_{ik} (see Ref. 11 , Vol. 2, p. 109). Let two flux lines, which are oriented parallel to the Z axis, cross the XY plane at the points $(r/2,0)$ and $(-r/2,0)$. It can be shown that the interaction force between the flux lines (per unit length) is

$$F_x = -\int_{-\infty}^{\infty} T_{xx}\, dy. \tag{4}$$

Since far from the centers of the vortices the field equations are linear, in performing the integration in Eq. (4) we assume that the solutions for $\mathbf{n}$ on the Y axis are a superposition of the solutions for solitary vortices [see Eqs. (2) and (3)]. Retaining in T_{xx} terms which are quadratic in $\theta \ll 1$ (3), we obtain the following expression for F_x in Eq. (4):

$$F_x = -4A^2(1+h)rK_1(r\sqrt{1+h}) = -2\sqrt{2\pi}A^2(1+h)^{3/4}r^{1/2}\exp(-r\sqrt{1+h}). \tag{5}$$

The last expression was obtained by expanding the modified Bessel function

$$K_1(k) = \int_0^{\infty} \exp(-k\sqrt{1+t^2})\, dt$$

for large arguments ($k \gg 1$) (see Ref. 14). Expression (5) describes the repulsion of identical particles. For particles with opposite parity, the sign in Eq. (5) is reversed.

Since in the present model the parity of the particles determines the character of their interaction (charge), CP invariance holds asymptotically. It should be kept in mind, however, that depending on the sign of k in Eq. (1) a vortex with one of the alternative directions of rotation of θ will be stable, i.e., the "antiparticle" is unstable. One can attempt to stabilize a vortex with the opposite sign by making its internal structure more complicated; for example, one can study vortex states by rotating θ through an angle which is a multiple of π. Such a vortex would have a high internal energy ("mass") compared to the vortex with the opposite charge (this reminds us how massive the proton is compared to an electron). We also note that the Lifshitz invariant in (1), which stabilizes localized states, does not affect the character of the interaction between localized states [see Eq. (5)].

3. Let us now consider from a more general standpoint the effect of interactions described by Lifshitz invariants on the stabilization of localized states. The form of the separate interactions in this case need not be specified. It is sufficient to separate in the functional of the system the terms which are quadratic (w_1) and linear (w_2) in the spatial derivatives, and also the terms which do not contain the spatial gradients (w_3). We now consider the states which are localized in D spatial coordinates and which are described by one configuration variable θ:

$$W^{(D)}=\int [w_1(\theta)+w_2(\theta)+w_3(\theta)]d^Dx. \tag{6}$$

Let $\theta_e^{(D)}(x)$ be the localized solution with dimension D. This assumes, specifically, that as $x\to\infty$, θ approaches an equilibrium value θ_0 and the integrals

$$I_i^{(D)}=\int w_i(\theta_e^{(D)})d^Dx \tag{7}$$

are finite. We introduce the new functions

$$\theta_a^{(D)}(x)=\theta_e^{(D)}(x/a). \tag{8}$$

The functions (8) were obtained by a radial deformation (stretching–compression) of the extremals $\theta_e^{(D)}(x)$. Substituting the function (8) into expression (6) makes it possible to calculate the function $W^{(D)}$ (a), which determines the change in the potentials (6) compared to their extremal values $W_0^{(D)}$ [calculated for the functions $\theta_e^{(D)}(x)$]:

$$W^{(D)}=I_1^{(D)}a^{D-2}+I_2^{(D)}a^{D-1}+I_3^{(D)}a^D. \tag{9}$$

The conditions for radial stability of the stationary localized solutions $\theta_e^{(D)}(x)$ are formulated as follows:

$$\left(\frac{dW^{(D)}}{da}\right)_{a=1}=(D-2)I_1^{(D)}+(D-1)I_2^{(D)}+DI_3^{(D)}=0, \tag{10}$$

$$\left(\frac{d^2W^{(D)}}{da^2}\right)_{a=1}=(D-2)(D-3)I_1^{(D)}+(D-1)(D-2)I_2^{(D)}+D(D-1)I_3^{(D)}>0. \tag{11}$$

The integral relations (10) determine the relative contribution of different energies to the total energy of the equilibrium localized states. For $I_2^{(D)}=0$ and $D=3$ relation (11) does not hold; i.e., for the functional (6) with $w_2=0$ the three-dimensional stationary localized states are energetically unstable. This result was obtained in Refs. 2 and 3 (Hobart–Derrick theorem) and later extended in Ref. 4 to the two-dimensional case.

The interactions described by the Lifshitz invariants do not change the condition of radial stability of one-dimensional localized states, but they are important for multidimensional systems. It follows from Eqs. (10) and (11) that two- and three-dimensional, radially stable, stationary localized states are possible in systems with an interactional functional of the general type (6). To avoid misunderstandings, it should be noted that in the present work we considered systems with a uniform ground state [this is the case if the energy w_2 in Eq. (6) is sufficiently low]. In this case it is sufficient to check the radial stability of the metastable localized states. If the energy w_2 is sufficiently high, however, then a modulated structure corresponds to the ground state. For example, for the potential (1) this happens for $k>1$. In studying the stability of the localized states in such systems it is necessary to consider disturbances of a more general type.[8]

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